
MATHEMATICAL
PHYSICS

Numerical Method for Solving the Microwave Tomography Problem of Reconstructing Inhomogeneities in a Cylindrical Body

Yu. G. Smirnov^{a,*}, M. Yu. Medvedik^{a,**}, and V. Yu. Martynova^{a,***}

^a Penza State University, Penza, 440026 Russian Federation

*e-mail: smirnovyug@mail.ru

**e-mail: _medv@mail.ru

***e-mail: lynxbax@mail.ru

Received July 9, 2025; revised July 9, 2025; accepted July 21, 2025

Abstract—In this work, a vector three-dimensional inverse problem of diffraction by a cylindrical body is solved based on a two-step method. The scatterer is filled with an inhomogeneous nonmagnetic dielectric material. The initial boundary value problem for the system of Maxwell's equations is reduced to a system of integro-differential equations. A numerical method for solving the equation of the first-kind, entering into the system, in special classes of functions is described. A distinctive feature of the proposed numerical method is its non-iterative nature; in addition, a good initial approximation is not required for the two-step method for solving the inverse problem. The calculation results are presented. It is shown that the two-step method is an efficient approach to solving vector problems of microwave tomography.

Keywords: three-dimensional vector inverse diffraction problem, permittivity reconstruction, integro-differential equation, two-step method, numerical method

DOI: 10.1134/S0965542525701222

INTRODUCTION

Problems of reconstructing the structure of objects from the results of near electromagnetic field measurements are used in a various areas of human activity, providing non-destructive analysis, high detail, and accuracy [1]. In industry, for example, these methods are used to diagnose microcracks, corrosion, and defects in aircraft components, power equipment, and composite materials, which helps prevent accidents and extend the service life of structures without their physical damage. In medicine and biology, near-field scanning technologies open up opportunities for studying cellular structures, tissues, and even teeth with submicron resolution, and miniature electromagnetic sensors are integrated into endoscopic systems for high-precision visualization of internal organs in real time [2–4].

Microelectronics and nanotechnology actively use near-field electromagnetic field methods to control the quality of semiconductor elements, nanostructures, and metamaterials, where traditional optics is limited by the diffraction limit. For example, scanning near-field microscopy allows obtaining images of material surfaces with nanometer resolution, which is critical for the development of new generations of processors and sensors [5, 6]. In geophysics, similar approaches are used to explore oil, gas, and mineral deposits, as well as to monitor soil stability during the construction of infrastructure facilities such as tunnels and high-rise buildings.

In telecommunications, these methods help optimize antenna design and improve radar systems, including synthetic aperture radars, which are used to detail the relief in navigation and mapping. The defense and environmental industries are also not left out: near-field electromagnetic technologies are used to scan baggage at airports to detect prohibited items, monitor radioactive contamination in the soil, or analyze the condition of agricultural crops.

The key advantages of these technologies are their non-destructive nature, the ability to operate in aggressive environments, and achieve a resolution that is inaccessible to classic optical systems. Modern development of machine learning algorithms and hybrid approaches combining the near electromagnetic field with acoustic or thermographic imaging significantly improves the reconstruction accuracy. Thus, near electromagnetic field analysis methods offer innovative solutions for complex problems.

Due to the widespread use of reconstruction problems in various technical fields, it is necessary to develop modern mathematical methods to improve the quality of diagnostic equipment.

It should be noted that methods for solving inverse problems have been actively developed over the decades and are presented in classical monographs [7–10]. The ideas and approaches presented in these works are also used in this article.

The two-step method, which is the basis of the numerical method for solving the inverse problem proposed in this paper, was presented in 2017 [11, 12]. Subsequently, it was successfully applied to solve two-dimensional [13], scalar [14–17], and vector electromagnetic [18, 19] problems. A theoretical justification of this method is given in [20].

STATEMENT OF THE FORWARD PROBLEM

We denote by $Q \subset \mathbb{R}^3$ a cylinder with boundary ∂Q . We assume that free space $\mathbb{R}^3 \setminus \bar{Q}$ is isotropic and homogeneous with constant ϵ_0 and μ_0 , where $\epsilon_0 > 0$ is the vacuum permittivity and $\mu_0 > 0$ is the vacuum permeability. The body Q is isotropic and inhomogeneous. The material of the scatterer Q is characterized by the value $\mu = \mu_0$ and a permittivity function $\epsilon(r)$.

For convenience, we introduce the relative permittivity $\epsilon = \epsilon \epsilon_0^{-1}$. We assume that each point $r \in \bar{Q}$ is characterized by a complex-valued function

$$\xi(r) = (\epsilon(r) - 1)^{-1}. \quad (1)$$

A source located at a point $r_s \in \mathbb{R}^3 \setminus \bar{Q}$ generates an electromagnetic wave $\mathbf{E}_0, \mathbf{H}_0 \in C^\infty(\mathbb{R}^3 \setminus r_s)$. The components of the incident field $\mathbf{E}_0, \mathbf{H}_0$ satisfy Maxwell's equations

$$\operatorname{curl} \mathbf{H}_0 = -i\omega \epsilon_0 \mathbf{E}_0, \quad \operatorname{curl} \mathbf{E}_0 = i\omega \mu_0 \mathbf{H}_0. \quad (2)$$

Let $\mathbf{E}_s, \mathbf{H}_s$ be the components of the scattered field; then, the total field is the sum of the incident and scattered fields:

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_s, \quad \mathbf{H} = \mathbf{H}_0 + \mathbf{H}_s. \quad (3)$$

The solution of the forward diffraction problem is the total electromagnetic field $\mathbf{E}, \mathbf{H}$ satisfying Maxwell's equations

$$\operatorname{curl} \mathbf{H} = -i\omega \epsilon \mathbf{E}, \quad \operatorname{curl} \mathbf{E} = i\omega \mu_0 \mathbf{H}, \quad (4)$$

conditions of continuity of the tangent components on the boundary of the heterogeneity region,

$$[\mathbf{E}_\tau]_{|\partial Q} = [\mathbf{H}_\tau]_{|\partial Q} = 0, \quad (5)$$

conditions of finite energy in any limited volume of space,

$$\mathbf{E}, \mathbf{H} \in \mathbf{L}_{2,\text{loc}}(\mathbb{R}^3), \quad (6)$$

and the Silver–Muller radiation conditions at infinity for the scattered field [21]:

$$\begin{aligned} \mathbf{H}_s \times \mathbf{e} - \mathbf{E}_s &= o(|\mathbf{x}|^{-1}), \quad \mathbf{E}_s = O(|\mathbf{x}|^{-1}), \\ \mathbf{E}_s \times \mathbf{e} + \mathbf{H}_s &= o(|\mathbf{x}|^{-1}), \quad \mathbf{H}_s = O(|\mathbf{x}|^{-1}), \quad |\mathbf{x}| \rightarrow \infty, \end{aligned} \quad (7)$$

where $\mathbf{e} = \mathbf{x}|\mathbf{x}|^{-1}$, $\mathbf{x} = (x_1, x_2, x_3)^T$, and

$$\mathbf{E}, \mathbf{H} \in C^2(\mathbb{R}^3 \setminus \partial Q) \cap C(\bar{Q}) \cap C(\mathbb{R}^3 \setminus Q). \quad (8)$$

Note that the limit relations in (7) are satisfied uniformly in all directions.

Definition 1. A solution of problem (2)–(7), $\mathbf{E}, \mathbf{H}$, satisfying conditions (8) is called classical.

We have the following results on the solvability of the direct diffraction problem [20].

Proposition 1. Let the relative permittivity satisfy the conditions

$$\epsilon(r) \in C^\infty(\bar{Q}), \quad \operatorname{Re}(\epsilon(r)) > 1, \quad \operatorname{Im}(\epsilon(r)) \geq 0, \quad r \in \bar{Q}, \quad (9)$$

and

$$\epsilon(r) = 1, \quad r \notin \bar{Q}.$$

In addition, everywhere $\mu = \mu_0$. Then, the diffraction problem (2)–(8) has at most one classical solution.

The boundary value problem (2)–(8) reduces to a system [22–24] consisting of an integro-differential equation over the region of inhomogeneity:

$$\mathbf{E}(r) - \left(k_0^2 + \operatorname{grad} \operatorname{div}\right) \int_Q G(r, r_0) (\xi(r_0))^{-1} \mathbf{E}(r_0) dr_0 = \mathbf{E}_0(r), \quad r \in Q, \quad (10)$$

and an integral representation of the field outside the body:

$$\mathbf{E}(r) = \mathbf{E}_0(r) + \left(k_0^2 + \operatorname{grad} \operatorname{div}\right) \int_Q G(r, r_0) (\xi(r_0))^{-1} \mathbf{E}(r_0) dr_0, \quad r \in \mathbb{R}^3 \setminus \bar{Q}, \quad (11)$$

where $k_0^2 = \omega^2 \epsilon_0 \mu_0$ is the wavenumber, the Green's function has the form

$$G(r, r_0) = \frac{e^{ik_0|r-r_0|}}{4\pi|r-r_0|},$$

and the incident field is determined by the Green's function with one fixed point $r_s \in \mathbb{R}^3 \setminus \bar{Q}$, which determines the position of the source,

$$\mathbf{E}_0(r) = G(r, r_s).$$

The magnetic field in the space $\mathbb{R}^3$ can be found by the formula

$$\mathbf{H} = \frac{1}{i\omega\mu_0} \operatorname{curl} \mathbf{E}. \quad (12)$$

We have the following statement about the equivalence of the boundary value problem (2)–(8) and the integro-differential equation.

Proposition 2. If the boundary value problem (2)–(8) has a classical solution $\mathbf{E}$, $\mathbf{H}$, then the vector function $\mathbf{E} \in C(\bar{Q}) \cap \mathbf{L}_2(Q)$ satisfies the integro-differential equation (11). Conversely, if $\mathbf{E} \in \mathbf{L}_2(Q)$ satisfies Eq. (11), then the boundary value problem (2)–(8) has a classical solution $\mathbf{E}$, $\mathbf{H}$, expressed by formulas (10)–(12).

Proposition 3. Let the relative permittivity satisfy conditions (9) at $r \in \bar{Q}$. Then, the operator

$$A\mathbf{E}(r) := \mathbf{E}(r) - \left(k_0^2 + \operatorname{grad} \operatorname{div}\right) \int_Q G(r, r_0) (\xi(r_0))^{-1} \mathbf{E}(r_0) dr_0$$

is continuously invertible in $\mathbf{L}_2(Q)$.

Propositions 1–3 were proved in [25, 26].

STATEMENT OF THE INVERSE PROBLEM

Consider the vector inverse problem of reconstructing the inhomogeneity of an isotropic dielectric cylinder Q located in a homogeneous three-dimensional space $\mathbb{R}^3$ characterized by a wavenumber $k_0 > 0$. For convenience, we used cylindrical coordinates $r = (\rho, \varphi, z)$, where $\rho \geq 0$, $0 \leq \varphi \leq 2\pi$, $z \in \mathbb{R}$. In the new notation, we have

$$\bar{Q} = \{r : 0 \leq \rho \leq R, 0 \leq \varphi \leq 2\pi, 0 \leq z \leq H\},$$

where R is the radius of the base and H is the height of the cylinder Q .

On the scatterer Q , we introduce a uniform grid with nodes

$$\rho_{i_1} = h_1 i_1, \quad \varphi_{i_2} = h_2 i_2, \quad z_{i_3} = h_3 i_3, \quad i_k = \overline{0, n_k}, \quad k = 1, 2, 3,$$

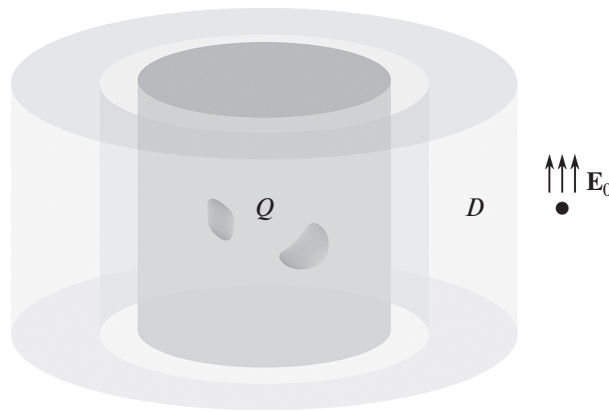


Fig. 1. Model of electromagnetic-field diffraction by a body Q with inhomogeneities.

where

$$h_1 = \frac{R}{n_1}, \quad h_2 = \frac{2\pi}{n_2}, \quad h_3 = \frac{H}{n_3}$$

are the grid steps and $n_k + 1$ is the number of grid nodes in the k th direction, $k = 1, 2, 3$.

According to the grid presented above, the cylinder Q consists of elementary cells

$$Q_{i_1 i_2 i_3} = \{r : \rho_{i_1} < \rho < \rho_{i_1+1}, \varphi_{i_2} < \varphi < \varphi_{i_2+1}, z_{i_3} < z < z_{i_3+1}\}, \quad i_k = \overline{0, n_k - 1}, \quad k = 1, 2, 3.$$

We introduce piecewise constant functions

$$\chi_{i_1 i_2 i_3}(r) = \begin{cases} 1, & r \in Q_{i_1 i_2 i_3}, \\ 0, & r \notin Q_{i_1 i_2 i_3}, \end{cases} \quad i_k = \overline{0, n_k - 1}, \quad k = 1, 2, 3. \quad (13)$$

The permeability of the body under study is assumed to be known and constant with the value $\mu_0 > 0$. The permittivity of the cylinder Q is unknown and is to be determined. We assume that the permittivity function $\varepsilon(r) = \varepsilon(r)\varepsilon_0$ is piecewise constant. To be more precise, we assume that

$$\varepsilon(r) = \{\varepsilon_{i_1 i_2 i_3}, \quad r \in Q_{i_1 i_2 i_3}, \quad i_k = \overline{0, n_k - 1}, \quad k = 1, 2, 3.$$

The permittivity at points on the boundaries $\partial Q_{i_1 i_2 i_3}$ can be defined as the limit of $\varepsilon(r)$ in one of the neighboring cells.

Using the basis functions (13), the permittivity $\varepsilon(r)$ at $r \in Q$ can be represented as

$$\varepsilon(r) = \sum_i \varepsilon_i(r) \chi_i(r),$$

where $i = (i_1 i_2 i_3)$ is a multi-index.

In free space outside the body Q , we select a bounded domain D . Thus, $\bar{D} \cap \bar{Q} = \emptyset$. We assume that the values of the total field at a fixed frequency $\omega > 0$ in the domain D are known.

An example of the geometry of the problem of diffraction of an electromagnetic field by an inhomogeneous body Q is shown in Fig. 1, where the black dot corresponds to the source and the outer cylindrical body with a ring-shaped base corresponds to the domain D .

Definition 2. The inverse diffraction problem consists in reconstructing the function $\varepsilon(r)$ in the region of inhomogeneity Q from the results of measuring the total field $\mathbf{E}(r)$ at observation points in the domain D ,

$$\left(k_0^2 + \text{grad div}\right) \int_Q G(r, r_0) (\xi(r_0))^{-1} \mathbf{E}(r_0) dr_0 = \mathbf{E}(r) - \mathbf{E}_0(r), \quad r \in D, \quad (14)$$

under the condition

$$\mathbf{E}(r) - \left(k_0^2 + \text{grad div}\right) \int_Q G(r, r_0) (\xi(r_0))^{-1} \mathbf{E}(r_0) dr_0 = \mathbf{E}_0(r), \quad r \in Q. \quad (15)$$

Next, we describe a two-step method for solving the inverse diffraction problem. First, we introduce a vector function $\mathbf{J}(r)$ on Q in the form

$$\mathbf{J}(r) := \frac{\mathbf{E}(r)}{\xi(r)},$$

where the function $\xi(r)$ is defined by formula (1). We assume that the condition $|\epsilon(r)| > 1$ is satisfied everywhere at $r \in \bar{Q}$. In the new notation, the representation of the field outside the scatterer (14) has the form

$$(k_0^2 + \text{grad div}) \int_Q G(r, r_0) \mathbf{J}(r_0) dr_0 = \mathbf{E}(r) - \mathbf{E}_0(r), \quad r \in D, \quad (16)$$

And the equation (15) for a point inside the body can be rewritten as

$$\xi(r) \mathbf{J}(r) - (k_0^2 + \text{grad div}) \int_Q G(r, r_0) \mathbf{J}(r_0) dr_0 = \mathbf{E}_0(r), \quad r \in Q. \quad (17)$$

The two-step method for reconstructing the permittivity of the cylinder Q is as follows. The first step is to use the known values of the incident field $\mathbf{E}_0(r)$ and the total field $\mathbf{E}(r)$ in the domain D outside the scatterer Q to find the values of the current $\mathbf{J}(r)$ in the body Q from Eq. (16). The second step is to explicitly calculate $\epsilon(r)$ in Q , using Eq. (17).

In [20], the uniqueness of the solution of such an inverse problem was proven.

NUMERICAL METHOD

Both the direct and inverse problems are solved by the collocation method. To implement this method, it is necessary to select the basis functions and collocation points.

Implementations of the method with basis functions of the form (13) were presented in [11, 13–15, 20]. For such basis functions, the carrier is one elementary cell. However, with this approach, it is impossible to introduce the divergence operation under the integral sign and “transfer” this operation to the basis function. Thus, we have a singular integro-differential equation. In this connection, there is a need to use special methods to take into account the singularities of the integrals.

In [22], basis functions of the following form were proposed:

$$v_i^{(\rho)}(r) = \begin{cases} 1 - \frac{|\rho - \rho_{i+1}|}{h_1}, & r \in Q_{i_1 i_2 i_3} \cup Q_{i_1+1 i_2 i_3}, \quad i_1 = \overline{0, n_1 - 2}, \quad i_k = \overline{0, n_k - 1}, \quad k = 2, 3, \\ 0, & r \notin Q_{i_1 i_2 i_3} \cup Q_{i_1+1 i_2 i_3}, \end{cases} \quad (18)$$

$$v_i^{(\varphi)}(r) = \begin{cases} 1 - \frac{|\varphi - \varphi_{i_2+1}|}{h_2}, & r \in Q_{i_1 i_2 i_3} \cup Q_{i_1 i_2+1 i_3}, \quad i_k = \overline{0, n_k - 1}, \quad k = 1, 2, 3, \\ 0, & r \notin Q_{i_1 i_2 i_3} \cup Q_{i_1 i_2+1 i_3}, \end{cases} \quad (19)$$

$$v_i^{(z)}(r) = \begin{cases} 1 - \frac{|z - z_{i_3+1}|}{h_3}, & r \in Q_{i_1 i_2 i_3} \cup Q_{i_1 i_2 i_3+1}, \quad i_3 = \overline{0, n_3 - 2}, \quad i_k = \overline{0, n_k - 1}, \quad k = 1, 2, \\ 0, & r \notin Q_{i_1 i_2 i_3} \cup Q_{i_1 i_2 i_3+1}, \end{cases} \quad (20)$$

where

$$Q_{i_1 n_2 i_3} = \{r : \rho_{i_1} < \rho < \rho_{i_1+1}, \varphi_{n_2} < \varphi < \varphi_{n_2} + h_2, z_{i_3} < z < z_{i_3+1}\}, \quad i_k = \overline{0, n_k - 1}, \quad k = 1, 3,$$

or $Q_{i_1 n_2 i_3} = Q_{i_1 0 i_3}$, $i_k = \overline{0, n_k - 1}$, $k = 1, 3$, which leads to the same result. With this choice of basis functions, we obtain integrals with only a weak singularity. It should be noted that three types of basis functions (18)–(20) are selected. In addition, the supports of these functions (18)–(20) are two adjacent cells.

Remark 1. Adding basis functions (19) with index $i_2 = n_2 - 1$ is possible due to the periodicity in φ .

The implementation of the collocation method requires calculating the following integrals:

$$(k_0^2 + \text{grad div}) \int_Q G(r, r_0) \tilde{\mathbf{J}}(r_0) dr_0,$$

where $\tilde{\mathbf{J}}$ is an approximate solution.

First, we introduce the following notation. Let $\boldsymbol{\rho}$, $\boldsymbol{\varphi}$, $\mathbf{z}$ be unit vectors in the corresponding directions. In this notation, the gradient operation in cylindrical coordinates for an arbitrary scalar function Φ at a point $r = (\rho, \varphi, z)$ has the following form:

$$\text{grad } \Phi = \frac{\partial \Phi}{\partial \rho} \boldsymbol{\rho} + \frac{1}{\rho} \frac{\partial \Phi}{\partial \varphi} \boldsymbol{\varphi} + \frac{\partial \Phi}{\partial z} \mathbf{z}. \quad (21)$$

Moreover, the divergence operation for an arbitrary vector function $\mathbf{F}$ has the form

$$\text{div } \mathbf{F} = \frac{1}{\rho} \frac{\partial(\rho \mathbf{F}_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial \mathbf{F}_\varphi}{\partial \varphi} + \frac{\partial \mathbf{F}_z}{\partial z}. \quad (22)$$

First of all, we find formulas for calculating the gradient. Using the formulas for the conversion from cylindrical coordinates to Cartesian coordinates

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad z = z,$$

we obtain

$$|r - r_0| = \sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\varphi - \varphi_0) + (z - z_0)^2},$$

where $r = (\rho, \varphi, z)$, $r_0 = (\rho_0, \varphi_0, z_0)$. Hence, using (21), we find

$$\text{grad } G(r, r_0) = g(|r - r_0|) \left(\frac{\partial |r - r_0|}{\partial \rho} \boldsymbol{\rho} + \frac{1}{\rho} \frac{\partial |r - r_0|}{\partial \varphi} \boldsymbol{\varphi} + \frac{\partial |r - r_0|}{\partial z} \mathbf{z} \right), \quad (23)$$

where

$$\frac{\partial |r - r_0|}{\partial \rho} = \frac{\rho - \rho_0 \cos(\varphi - \varphi_0)}{|r - r_0|}, \quad \frac{\partial |r - r_0|}{\partial \varphi} = \frac{\rho \rho_0 \sin(\varphi - \varphi_0)}{|r - r_0|}, \quad \frac{\partial |r - r_0|}{\partial z} = \frac{z - z_0}{|r - r_0|},$$

and the function $g(t)$ has the form

$$g(t) = \left(ik_0 - \frac{1}{t} \right) \frac{e^{ik_0 t}}{4\pi t}.$$

Next, we find formulas for calculating the divergence using vector basis functions

$$\begin{aligned} \mathbf{v}_i^{(\rho)}(r) &= v_i^{(\rho)}(r) \boldsymbol{\rho}(r), \quad i_1 = \overline{0, n_1 - 2}, \quad i_k = \overline{0, n_k - 1}, \quad k = 2, 3, \\ \mathbf{v}_i^{(\varphi)}(r) &= v_i^{(\varphi)}(r) \boldsymbol{\varphi}(r), \quad i_k = \overline{0, n_k - 1}, \quad k = 1, 2, 3, \\ \mathbf{v}_i^{(z)}(r) &= v_i^{(z)}(r) \mathbf{z}(r), \quad i_3 = \overline{0, n_3 - 2}, \quad i_k = \overline{0, n_k - 1}, \quad k = 1, 2. \end{aligned} \quad (24)$$

Now we find the divergence from the basis functions (24) using formulas (18)–(20) and (22). We obtain the following formulas for calculating the divergence from the basis functions (24) within their supports:

$$\begin{aligned} \text{div } \mathbf{v}_i^{(\rho)}(r) &= \frac{1}{\rho} v_i^{(\rho)}(r) - \frac{\text{sign}(\rho - \rho_{i_1+1})}{h_1}, \quad i_1 = \overline{0, n_1 - 2}, \quad i_k = \overline{0, n_k - 1}, \quad k = 2, 3, \\ \text{div } \mathbf{v}_i^{(\varphi)}(r) &= -\frac{\text{sign}(\varphi - \varphi_{i_2+1})}{\rho h_2}, \quad i_k = \overline{0, n_k - 1}, \quad k = 1, 2, 3, \\ \text{div } \mathbf{v}_i^{(z)}(r) &= -\frac{\text{sign}(z - z_{i_3+1})}{h_3}, \quad i_3 = \overline{0, n_3 - 2}, \quad i_k = \overline{0, n_k - 1}, \quad k = 1, 2, \end{aligned} \quad (25)$$

and at points not belonging to the carriers, the divergences are equal to zero. Due to the specificity of the basis functions, the divergences from them are found in a standard way, and not as generalized functions, and are piecewise continuous.

It is convenient to seek an approximate solution $\tilde{\mathbf{J}}$ in the form

$$\tilde{\mathbf{J}}(r) = \tilde{\mathbf{J}}^{(\rho)}(r) + \tilde{\mathbf{J}}^{(\varphi)}(r) + \tilde{\mathbf{J}}^{(z)}(r), \quad (26)$$

where $\tilde{\mathbf{J}}^{(\rho)}$, $\tilde{\mathbf{J}}^{(\varphi)}$, and $\tilde{\mathbf{J}}^{(z)}$ are linear combinations of the corresponding basis functions with unknown coefficients:

$$\begin{aligned}\tilde{\mathbf{J}}^{(\rho)}(r) &= \sum_i J_i^{(\rho)} \mathbf{v}_i^{(\rho)}(r), \quad i_1 = \overline{0, n_1 - 2}, \quad i_k = \overline{0, n_k - 1}, \quad k = 2, 3, \\ \tilde{\mathbf{J}}^{(\varphi)}(r) &= \sum_i J_i^{(\varphi)} \mathbf{v}_i^{(\varphi)}(r), \quad i_k = \overline{0, n_k - 1}, \quad k = 1, 2, 3, \\ \tilde{\mathbf{J}}^{(z)}(r) &= \sum_i J_i^{(z)} \mathbf{v}_i^{(z)}(r), \quad i_3 = \overline{0, n_3 - 2}, \quad i_k = \overline{0, n_k - 1}, \quad k = 1, 2.\end{aligned}\quad (27)$$

We emphasize an important property that the normal component of the current $\tilde{\mathbf{J}}$ vanishes at the smoothness points of the boundary ∂Q . In other words, $\tilde{\mathbf{J}}(r) \cdot \mathbf{v}(r) = 0$, where $\mathbf{v}(r)$ is the outer normal to the boundary at the point $r \in \partial Q$. It should be noted that the introduced basis functions have the approximative property in the space $L_2(Q)$.

The next step will require a formula for calculating the divergence of the volume integral. In [22], it was proven that

$$\operatorname{div}_r \int_Q G(r, r_0) \tilde{\mathbf{J}}(r_0) dr_0 = \int_Q G(r, r_0) \operatorname{div} \tilde{\mathbf{J}}(r_0) dr_0 \quad (28)$$

is satisfied for any $r \in \mathbb{R}^3$. Since the gradient and divergence operations are invariant with respect to coordinate systems, formula (28) can be obtained in Cartesian coordinates and applied in cylindrical ones.

In addition, we calculate

$$\operatorname{grad}_r \operatorname{div}_r \int_Q G(r, r_0) \tilde{\mathbf{J}}(r_0) dr_0 = \int_Q \operatorname{grad}_r G(r, r_0) \operatorname{div} \tilde{\mathbf{J}}(r_0) dr_0$$

as an improper absolutely convergent integral, because the kernel has a singularity of order $O(|r - r_0|^{-2})$ at $|r - r_0| \rightarrow 0$ (weak singularity). Using (23) and (25) we obtain $\operatorname{grad}_x G(r, r_0)$ and $\operatorname{div} \tilde{\mathbf{J}}(r_0)$, respectively.

Finally, we get

$$\begin{aligned}(k_0^2 + \operatorname{grad}_r \operatorname{div}_r) \int_Q G(r, r_0) \tilde{\mathbf{J}}(r_0) d\rho_0 &= k_0^2 \int_0^R \int_0^{2\pi} \int_0^H G(\rho, \varphi, z, \rho_0, \varphi_0, z_0) \tilde{\mathbf{J}}(\rho_0, \varphi_0, z_0) \rho_0 d\rho_0 d\varphi_0 dz_0 \\ &+ \int_0^R \int_0^{2\pi} \int_0^H \operatorname{grad} G(\rho, \varphi, z, \rho_0, \varphi_0, z_0) \operatorname{div} \tilde{\mathbf{J}}(\rho_0, \varphi_0, z_0) \rho_0 d\rho_0 d\varphi_0 dz_0.\end{aligned}$$

In [22], three types of collocation points were proposed for the forward problem in the form

$$\begin{aligned}r_i^{(\rho)} &= \left(\rho_{i_1} + h_1, \varphi_{i_2} + \frac{h_2}{2}, z_{i_3} + \frac{h_3}{2} \right), \quad i_1 = \overline{0, n_1 - 2}, \quad i_k = \overline{0, n_k - 1}, \quad k = 2, 3, \\ r_i^{(\varphi)} &= \left(\rho_{i_1} + \frac{h_1}{2}, \varphi_{i_2} + h_2, z_{i_3} + \frac{h_3}{2} \right), \quad i_k = \overline{0, n_k - 1}, \quad k = 1, 2, 3, \\ r_i^{(z)} &= \left(\rho_{i_1} + \frac{h_1}{2}, \varphi_{i_2} + \frac{h_2}{2}, z_{i_3} + h_3 \right), \quad i_3 = \overline{0, n_3 - 2}, \quad i_k = \overline{0, n_k - 1}, \quad k = 1, 2.\end{aligned}\quad (29)$$

In this case, the collocation points correspond uniquely to the basis functions and the number of nodes is equal to the number of basis functions, which leads to a square matrix of the system of linear algebraic equations

$$M\mathbf{c} = -\mathbf{f}$$

in the collocation method, where $\mathbf{c}$ is the vector of unknown coefficients

$$\begin{aligned}J_i^{(\rho)}, \quad i_1 &= \overline{0, n_1 - 2}, \quad i_k = \overline{0, n_k - 1}, \quad k = 2, 3, \\ J_i^{(\varphi)}, \quad i_k &= \overline{0, n_k - 1}, \quad k = 1, 2, 3, \\ J_i^{(z)}, \quad i_3 &= \overline{0, n_3 - 2}, \quad i_k = \overline{0, n_k - 1}, \quad k = 1, 2,\end{aligned}\quad (30)$$

$\mathbf{f}$ is the vector of values of the incident field at the collocation points (29). The matrix M is a block 3×3 matrix and can be represented as $M = A - B$, where the elements of the matrix A are defined as follows:

$$a_{ij}^{(\alpha\beta)} = k_0^2 \int_0^R \int_0^{2\pi} \int_0^H G(r_i^{(\alpha)}; \rho_0, \varphi_0, z_0) v_j^{(\beta)}(\rho_0, \varphi_0, z_0) \rho_0 d\rho_0 d\varphi_0 dz_0 \\ + \int_0^R \int_0^{2\pi} \int_0^H \text{grad}^{(\alpha)} G(r_i^{(\alpha)}; \rho_0, \varphi_0, z_0) \text{div } v_j^{(\beta)}(\rho_0, \varphi_0, z_0) \rho_0 d\rho_0 d\varphi_0 dz_0, \quad (31)$$

and the elements of the matrix B have the form

$$b_{ij}^{(\alpha\beta)} = \xi(r_i^{(\alpha)}) v_j^{(\beta)}(r_i^{(\alpha)}),$$

$\mathbf{i} = (i_1 i_2 i_3)$ and $\mathbf{j} = (j_1 j_2 j_3)$ are multi-indexes, $\alpha, \beta = \rho, \varphi, z$, and $\text{grad}^{(\alpha)}$ are the corresponding components of the gradient $\text{grad} = (\text{grad}^{(\rho)}, \text{grad}^{(\varphi)}, \text{grad}^{(z)})$. In the chosen notation, the elements of the vectors $\mathbf{c}$ and $\mathbf{f}$ have the form

$$c_i^{(\alpha)} = J_i^{(\alpha)}, \quad f_i^{(\alpha)} = \mathbf{E}_0(r_i^{(\alpha)}).$$

It should be noted that the upper bounds of the indices i_k , $k = 1, 2, 3$, in the multi-index $\mathbf{i}$ depend on the direction α . Similarly, the upper bounds of the indices j_k , $k = 1, 2, 3$, in the multi-index $\mathbf{j}$ depend on the direction β .

Thus, numerically solving the forward problem, we find the values of the current $\tilde{\mathbf{J}}$ on Q , using (26) and (27). The forward problem allows us to model the values of the near field for further solving the inverse problem. Using $\tilde{\mathbf{J}}$, we calculate the values of the near field using the formula

$$\mathbf{E}(\tilde{r}_i^{(\alpha)}) = f_i^{(\alpha)} + k_0^2 \int_0^R \int_0^{2\pi} \int_0^H G(\tilde{r}_i^{(\alpha)}; \rho_0, \varphi_0, z_0) \tilde{\mathbf{J}}(\rho_0, \varphi_0, z_0) \rho_0 d\rho_0 d\varphi_0 dz_0 \\ + \int_0^R \int_0^{2\pi} \int_0^H \text{grad}^{(\alpha)} G(\tilde{r}_i^{(\alpha)}; \rho_0, \varphi_0, z_0) \text{div } \tilde{\mathbf{J}}(\rho_0, \varphi_0, z_0) \rho_0 d\rho_0 d\varphi_0 dz_0$$

at the selected observation points $\tilde{r}_i^{(\alpha)} \in D$, $\alpha = \rho, \varphi, z$. As before, the upper bounds of the indices i_k , $k = 1, 2, 3$, in the multi-index $\mathbf{i}$ depend on the direction α . The observation points $\tilde{r}_i^{(\rho)}, \tilde{r}_i^{(\varphi)}, \tilde{r}_i^{(z)} \in D$ are located in several layers around the lateral surface of the cylinder Q at a small distance.

Next, the problem of reconstructing the heterogeneity of the body, as mentioned above, is solved in two stages. At the first stage, we obtain the system

$$A\mathbf{c} = \mathbf{q} - \tilde{\mathbf{f}}, \quad (32)$$

where the elements of the matrix A have the form (31), the elements of the vectors $\mathbf{q}$ and $\tilde{\mathbf{f}}$ are defined as

$$q_i^{(\alpha)} = \mathbf{E}(\tilde{r}_i^{(\alpha)}), \quad \tilde{f}_i^{(\alpha)} = \mathbf{E}_0(\tilde{r}_i^{(\alpha)}),$$

and, as in the forward problem, $\mathbf{c}$ is a vector of unknown coefficients (30). The matrix A is a block matrix of size 3×3 .

At the second step, we reconstruct the value of the function ξ at the collocation points (29) using the recalculation formula

$$\xi(r_i^{(\alpha)}) = \left(\sum_{\beta} \sum_{\mathbf{j}} a_{ij}^{(\alpha\beta)} J_{\mathbf{j}}^{(\beta)} + f_i^{(\alpha)} \right) \left(\sum_{\beta} \sum_{\mathbf{j}} v_{\mathbf{j}}^{(\beta)}(r_i^{(\alpha)}) J_{\mathbf{j}}^{(\beta)} \right)^{-1}, \quad (33)$$

where $\alpha, \beta = \rho, \varphi, z$.

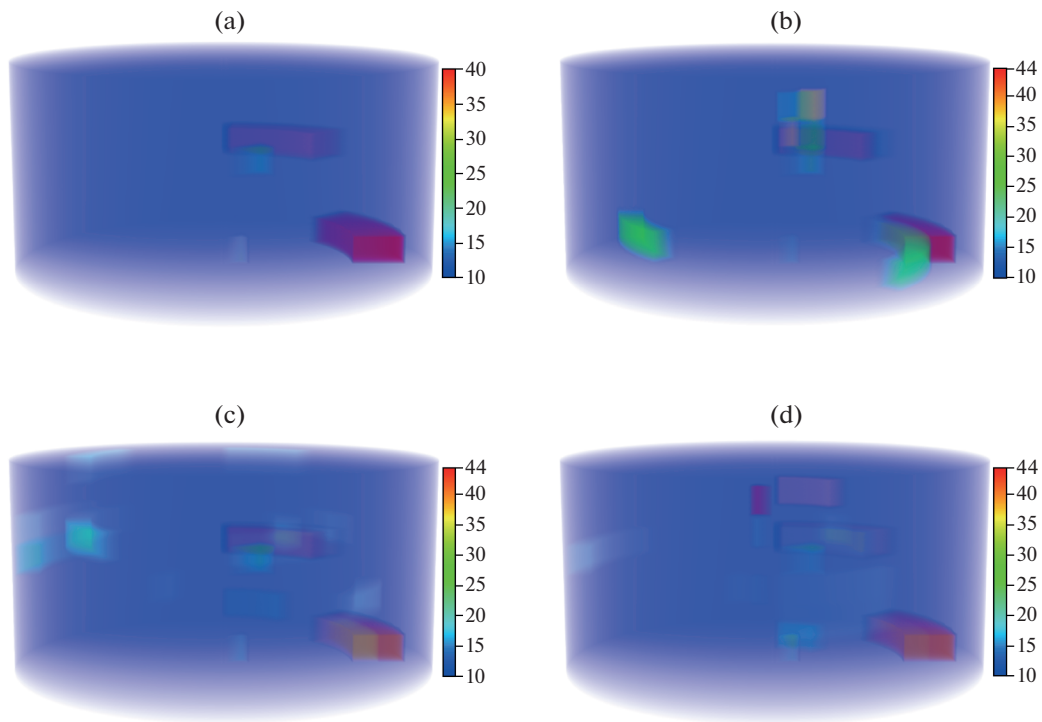


Fig. 2. Absolute value of permittivity in the problem of reconstruction by two-step method (32) and (33) with a 1% error: (a) original structure of the body Q ; (b) body structure reconstructed at a frequency of 5.76 GHz; (c) body structure reconstructed at a frequency of 5.87 GHz; (d) body structure reconstructed at a frequency of 5.87 GHz with a rotation of the source around the axis of the cylinder Q by 60° .

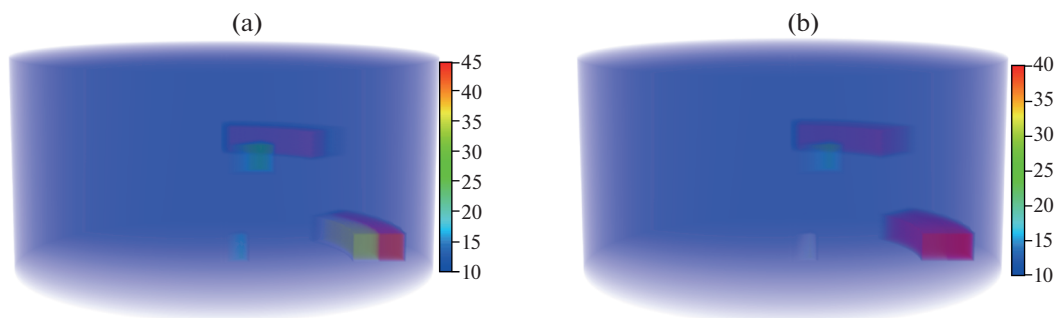


Fig. 3. Reconstructed structure of the body Q with a 1% error: (a) filtering by data at different frequencies; (b) filtering by data for different source positions.

NUMERICAL RESULTS

The initial structure of the inhomogeneous cylinder Q is shown in Fig. 2a. The absolute values of the complex-valued permittivity function in the problem of reconstruction by the two-step method (32) and (33) are shown in Figs. 2–6. The results are visualized using the freeware ParaView program. The calculations used the parameters described below. The radiation source is located at a distance from the lateral surface outside the body and the observation points, as shown in Fig. 1. The radius R of the cylinder bases is 0.05 m. The height H of the cylinder is 0.05 m. For the body Q under study, a uniform computational grid in a cylindrical coordinate system was constructed. The number of divisions of the grid in each direction is 8. The size of each matrix is 1408. The numerical experiments were carried out at different frequencies in the microwave range.

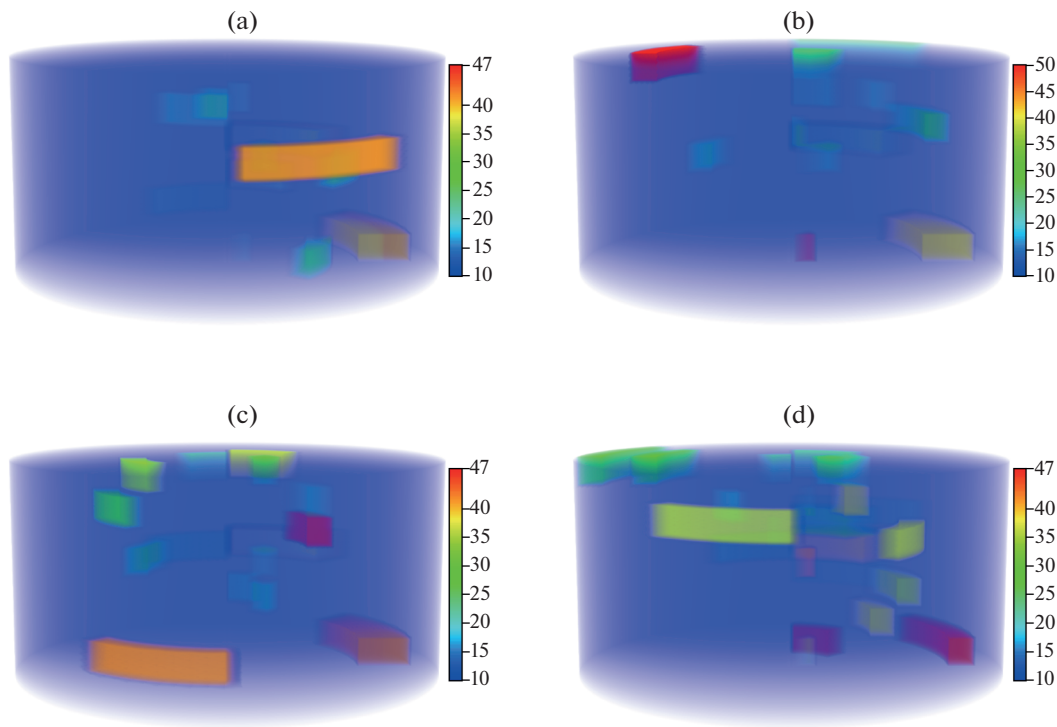


Fig. 4. Absolute value of permittivity in the problem of reconstruction by two-step method (32) and (33) with a 3% error: (a) body structure reconstructed at a frequency of 5.80 GHz; (b) body structure reconstructed at a frequency of 5.88 GHz; (c) body structure reconstructed at a frequency of 5.88 GHz with a rotation of the source around the axis of the cylinder Q by 60° ; (d) body structure reconstructed at a frequency of 5.87 GHz with a rotation of the source around the axis of the cylinder Q by 120° .

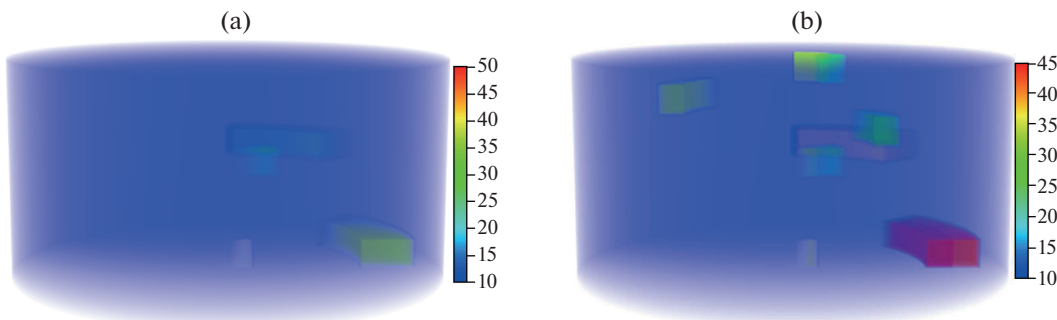


Fig. 5. Reconstructed structure of the body Q with a 3% error: (a) filtering by data at different frequencies; (b) filtering by data for different source positions.

Into the calculated near field used to solve the inverse problem shown in Figs. 2–6, we introduced white noise. The percentage of error is indicated in the figure captions. For correct reconstruction of the inhomogeneities of the body, we used preprocessing of the initial data and filtering of the results.

At the preprocessing stage, the initial data on the near field are cleared. In this case, the error is randomly introduced several times into the same calculated values, which are then averaged. This corresponds to averaging the data from the device, which are measured several times under the same conditions. This approach allows us to reduce the measurement error.

The procedure for filtering the calculation results consists in finding and removing “artifacts.” To do this, the problem is solved several times under different conditions. It is possible to change the frequency and position of the source. Then, the solution results are compared with each other. Real inhomogeneities

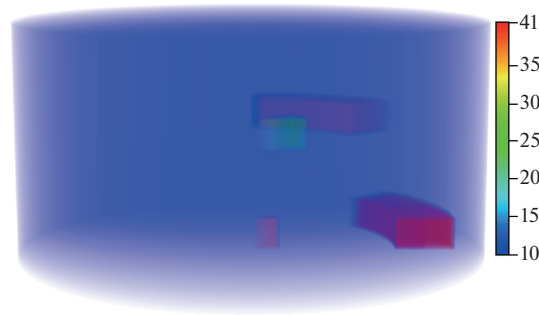


Fig. 6. Reconstructed structure of the body Q at the final stage of filtering with a 3% error.

remain in their original places, and artifacts are moved to different positions. This property is used by the filtering procedure. By comparing the corresponding solutions of the problem at different frequencies, non-repeating inhomogeneities are discarded. Inhomogeneities that remain in constant positions are averaged.

Figures 2b–2d demonstrate the reconstructed values of the piecewise constant function of the permittivity of the body Q , obtained using the two-step method (32) and (33), where white noise at a level of 1% is introduced into the calculated near field. Figures 2b and 2c are plotted for different frequencies. They demonstrate that the position of the artifacts changes and the inhomogeneities remain in the desired cells. A similar situation is observed in Figs. 2c and 2d, where the position of the radiation source was changed. The result of using the filtering procedure described above is shown in Fig. 3. Figure 3a shows the result of reconstructing the structure of the body Q with filtering at different frequencies. Given the results at different frequencies, it was possible to filter out all the artifacts. Figure 3b shows the result of reconstructing the structure of the body Q with filtering for different positions of the source. Due to rotations, it was possible to filter out false inhomogeneities, and the absolute values of the true inhomogeneities in this case are closer to the original structure shown in Fig. 2a. The maximum relative error in the comparison of the reconstructed final results presented in Fig. 3 with the initial data does not exceed 5%. The numerical results obtained show that, with a 1% error in the initial data, two measurements in different configurations are sufficient for complete reconstruction of the body structure.

Figure 4 shows the reconstructed values of the piecewise constant function of the permittivity of the body Q with an introduced white noise at a level of 3% into the calculated near field. Figures 4a and 4b were plotted for different frequencies, and, in Figs. 4c and 4d, the position of the radiation source was changed. Due to the increase in the error, the number of artifacts has grown and the values of some of the true inhomogeneities have approached the background values.

Figure 5 shows the intermediate stages of filtering the obtained data. As a result of applying the filtering procedure using calculations at three different frequencies in the range from 5.8 GHz to 5.88 GHz, it was possible to reconstruct the structure of the original object after introducing an error at a level of 3%. This solution is shown in Fig. 5a. Taking into account the results at different frequencies, it was possible to completely get rid of all artifacts and find all necessary inhomogeneities, but all their absolute values at this stage of filtering differ from the original ones. The results of reconstructing the body structure as a result of applying the filtering procedure using calculations at different positions of the radiation source are slightly worse. This solution is shown in Fig. 5b. Using rotations, it was possible to filter out most of the artifacts and not lose the true inhomogeneities.

The final data reconstruction with a 3% error is shown in Fig. 6, where the filtering based on all the data (both with a change in frequency and with a change in the source position) was used. Using this combined approach, it was possible to completely get rid of artifacts and reconstruct inhomogeneities with values close to the original ones with a 3% error in the initial data. The maximum relative error in comparing the reconstructed final results presented in Fig. 6 with the original data does not exceed 4%. Thus, the numerical results show that introducing an error into the initial data even at a level of 3% at the stage of measuring the near field does not significantly affect the final result of reconstructing the inhomogeneities in the body.

The matrix A used in solving the inverse problem is ill-conditioned. Therefore, we used the regularization procedure [27] in the form

$$(A^*A + \gamma I)\mathbf{c} = A^*(\mathbf{g} - \tilde{\mathbf{f}}),$$

where I is the identity matrix. The value of the parameter γ is selected taking into account the specifics of the matrix A by selection in accordance with Tikhonov's method proposed in the book [27, p. 78].

In this article, solutions of the direct problem are used to simulate the field values at observation points in the domain D . However, in practice, the actual field values at observation points are measured using special devices.

Calculations of the inverse acoustic problem by the two-step method for a hemisphere and a cylinder are given in [12, 16, 17]. Solutions of direct and inverse electrodynamic problems by the two-step method for structures in the form of a rectangular parallelepiped are presented in [18, 19].

CONCLUSIONS

In this paper, we have studied the problem of reconstructing the structure of a cylindrical body from the results of measuring the electromagnetic field outside the body. The problem is solved in cylindrical coordinates and at the electrodynamic level of rigor. A two-step method for solving the inverse problem has been implemented. To improve the properties of the resulting matrix of the system of linear algebraic equations, a numerical method for solving the inverse diffraction problem was proposed, based on the collocation method with piecewise linear basis functions along one of the coordinates. This approach allows using integration by parts to transfer the divergence operation to basis functions, which leads to the need to calculate integrals only with a weak singularity. For the two-step method described in the paper, the regularization procedure is performed at the stage of solving the system of linear algebraic equations.

The following advantages of the two-step method should be emphasized. Firstly, it is stable with respect to errors. Secondly, this method does not require any initial approximations, which differs from iterative approaches. In addition, computational experiments have shown that increasing the frequency allows detecting inhomogeneities of smaller sizes. Thus, in essence, a technology for solving the inverse problem of reconstructing the permittivity of an inhomogeneous body from the results of measurements of the electromagnetic field in the near zone has been proposed, which includes

- preprocessing of input data (averaging several measurement results);
- solving the inverse problem by a two-step method;
- repeatedly solving the problem with a change in the source position and at other frequencies in order to detect and remove artifacts;
- averaging the results of solving inverse problems.

In conclusion, it should be noted that the proposed two-step method can be used to solve a wide range of applied problems, including medical diagnostics.

FUNDING

This work was supported by the Ministry of Education and Science of the Russian Federation within the framework of a State assignment (grant no. 124020200015-7).

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

REFERENCES

1. V. G. Romanov, *Inverse Problems in Mathematical Physics* (Akad. Nauk SSSR, Moscow, 1984) [in Russian].
2. P. M. Meaney, D. Goodwin, A. H. Golnabi, T. Zhou, M. Pallone, S. D. Geimer, G. Burke, and K. D. Paulsen, "Clinical microwave tomographic imaging of the calcaneus: A first-in-human case study of two subjects," *IEEE Trans. Biomed. Eng.* **59** (12), 3304–3313 (2012).
3. A. M. Kamal, T. Sakorikar, U. M. Pal, and H. J. Pandya, "Engineering approaches for breast cancer diagnosis: A review," *IEEE Rev. Biomed. Eng.* **16**, 687–705 (2023).
4. V. Cherepenin, A. Karpov, A. Korjnevsky, V. Kornienko, A. Mazaletskaya, D. Mazourov, and D. Meister, "A 3D electrical impedance tomography (EIT) system for breast cancer detection," *Physiol. Meas.* **22** (1), 9–18 (2001).

5. D. Tranchida, J. Diaz, P. Schon, H. Schonherr, and G. J. Vancso, “Scanning near-field EllipsometryMicroscopy: Imaging nanomaterials with resolution below the diffraction limit,” *Nanoscale* **3**, 233–239 (2011).
6. M. Fleischer, “Near-field scanning optical microscopy nanoprobe,” *Nanotechnol. Rev.* **1** (4), 313–338 (2012).
7. M. M. Lavrent’ev, V. G. Romanov, and S. P. Shishatskii, *Ill-Posed Problems of Mathematical Physics and Analysis* (Nauka, Moscow, 1980; Am. Math. Soc., Providence, R.I., 1986).
8. A. B. Bakushinsky and A. V. Goncharsky, *Iterative Methods for Solving Ill-Posed Problems* (Nauka, Moscow, 1989) [in Russian].
9. A. N. Tikhonov, A. V. Goncharsky, V. V. Stepanov, and A. G. Yagola, *Numerical Methods for the Solution of Ill-Posed Problems* (Nauka, Moscow, 1990; Kluwer Academic, Dordrecht, 1995).
10. S. I. Kabanikhin, *Inverse and Ill-Posed Problems: Theory and Applications* (Sibirskoe Nauchnoe, Novosibirsk, 2008; De Gruyter, Berlin, 2011).
11. M. Y. Medvedik, Y. G. Smirnov, and A. A. Tsupak, “Two-step method for solving inverse problem of diffraction by an inhomogeneous body,” *Nonlinear and Inverse Problems in Electromagnetics (PIERS 2017)*, Ed. by L. Beilina and Y. Smirnov, *Springer Proceedings in Mathematics and Statistics* (Springer, 2018), Vol. 243, pp. 83–92.
12. R. O. Evstigneev and M. Y. Medvedik, “Reconstruction of inhomogeneity parameters by measurements of near field outside the body,” *Nonlinear and Inverse Problems in Electromagnetics (PIERS 2017)*, Ed. by L. Beilina and Y. Smirnov, *Springer Proceedings in Mathematics and Statistics* (Springer, 2018), Vol. 243, pp. 29–38.
13. M. Y. Medvedik, Y. G. Smirnov, and A. A. Tsupak, “The two-step method for determining a piecewise-continuous refractive index of a 2D scatterer by near field measurements,” *Inverse Probl. Sci. Eng.* **28** (3), 427–447 (2020).
14. M. Y. Medvedik, Y. G. Smirnov, and A. A. Tsupak, “Non-iterative two-step method for solving scalar inverse 3D diffraction problem,” *Inverse Probl. Sci. Eng.* **28** (1), 1474–1492 (2020).
15. Y. G. Smirnov and A. A. Tsupak, “Direct and inverse scalar scattering problems for the Helmholtz equation in $\mathbb{R}^m$,” *J. Inverse Ill-Posed Probl.* **30** (1), 101–116 (2022).
16. R. O. Evstigneev and M. Y. Medvedik, “Reconstruction of inhomogeneities in a hemisphere from the field measurements,” *Lobachevskii J. Math.* **40** (10), 1653–1659 (2019).
17. A. O. Lapich and M. Y. Medvedik, “Microwave tomography method for solving the inverse problem on cylindrical bodies,” *Tech. Phys. Lett.* **50**, 360–364 (2024).
18. A. O. Lapich and M. Y. Medvedik, “Method of generalized and combined computational grids for restoration of the parameters of inhomogeneities of a body based on the results of measurements of the electromagnetic field,” *Math. Models Comput. Simul.* **16** (6), 806–813 (2024).
19. A. O. Lapich and M. Yu. Medvedik, “Method for reconstruction the parameters of body inhomogeneities from the results of electromagnetic field measurements,” *Lobachevskii J. Math.* **45** (10), 4740–4747 (2024).
20. M. Y. Medvedik, Y. G. Smirnov, and A. A. Tsupak, “Inverse vector problem of diffraction by inhomogeneous body with a piecewise smooth permittivity,” *J. Inverse Ill-Posed Probl.* **32** (3), 453–465 (2024).
21. D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory* (Springer, Berlin, 2013).
22. E. Smolkin, Y. Smirnov, and M. Snegur, “Solution of the vector three-dimensional inverse problem on an inhomogeneous dielectric hemisphere using a two-step method,” *Computation* **12** (11), 213 (2024).
23. A. B. Samokhin and Yu. V. Shestopalov, *Integral Equations and Iteration Methods in Electromagnetic Scattering* (De Gruyter, Berlin, 2001).
24. Yu. G. Smirnov and A. A. Tsupak, *Diffraction of Acoustic and Electromagnetic Waves by Screens and Inhomogeneous Solids: Mathematical Theory* (RU-SCIENCE, Moscow, 2018).
25. Yu. G. Smirnov, A. A. Tsupak, and D. V. Valovik, “On the volume singular integro-differential equation for the electromagnetic diffraction problem,” *Appl. Anal. Int. J.* **96** (2), 173–189 (2017).
26. Yu. G. Smirnov and A. A. Tsupak, “Existence and uniqueness theorems in electromagnetic diffraction on systems of lossless dielectrics and perfectly conducting screens,” *Appl. Anal. Int. J.* **96** (8), 1326–1341 (2017).
27. A. N. Tikhonov and V. Ya. Arsenin, *Solutions of Ill-Posed Problems* (Halsted, New York, 1977).

Translated by E. Chernokozhin

Publisher’s Note. Pleiades Publishing remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
AI tools may have been used in the translation or editing of this article.